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1984 J. Phys. A: Math. Gen. 17 1397

(http://iopscience.iop.org/0305-4470/17/7/007)

J. Phys. A: Math. Gen. 17 (1984) 1397–1402. Printed in Great Britain

Branching rules for $E_8 \downarrow SO_{16}$

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Received 2 November 1983

Abstract. Branching rules for $E_8 \downarrow SO_{16}$ are derived from those of $E_8 \downarrow SU_2 \times (E_7 \downarrow SU_8)$ and $SO_{16} \downarrow U_1 \times SU_8$ by noting that both group chains share a common $U_1 \times SU_8$ subgroup. Additional branching rules are deduced from Kronecker products of E_8 and SO_{16} irreps. In each case the rules distinguish unambiguously between conjugate irreps of SO_{16}. An alternative labelling scheme for the irreps of SO_{2k} based on its maximal $U_1 \times SU_k$ subgroup is outlined.

1. Introduction

The exceptional groups continue to interest physicists developing grand unification and supergravity theories (Slansky 1981). In recent years considerable progress has been made in calculating the properties of the exceptional group irreps. Extensive tabulations of Kronecker products and branching rules have been given (Wybourne and Bowick 1977, Wybourne 1979, McKay and Patera 1981) for the exceptional groups. King and Al-Qubanchi (1981a) following upon Wybourne and Bowick (1977) have developed a natural labelling scheme for the irreps of the exceptional groups based upon their classical maximal subgroups and indicated their relationship to the corresponding Dynkin labelled irreps.

In the particular case of E_8, King and Al-Qubanchi based their labelling of the irreps of E_8 upon those of the maximal subgroup SO_{16} and went on (King and Al-Qubanchi 1981b) to deduce from weight multiplicity considerations some $E_8 \downarrow SO_{16}$ branching rules.

Wybourne and Bowick (1977) gave branching rules for $E_8 \downarrow SU_9$ and $E_8 \downarrow SU_2 \times E_7$ which were further extended by Wybourne (1979). Their methods involved the use of dimensional and Dynkin indexes and are inappropriate for the case of $E_8 \downarrow SO_{16}$ since SO_{16} possesses conjugate pairs of irreps that are not distinguished by dimensions or Dynkin indexes. In this paper a $U_1 \times SU_8$ subgroup common to two group chains is exploited to give $E_8 \downarrow SO_{16}$ branching rules unequivocally from those found for $E_8 \downarrow SU_2 \times E_7$. Many of the notational details have been developed in a series of recent papers (King *et al* 1981, Black *et al* 1983, Black and Wybourne 1983) and the reader is referred to these papers for essential details.

2. Labelling SO_{2k} irreps

Wybourne and Bowick (1977) pointed out the desirability of labelling the irreps of an exceptional group G in terms of one of its maximal subgroups H, an idea extended by King and Al-Qubanchi (1981a). With such a labelling system established we are assured that under the restriction of the irrep λ under $G\downarrow H$ we have

$$\lambda\downarrow\lambda+\ldots. \tag{1}$$

The k fundamental irreps of SO_{2k} are traditionally labelled in the Cartan–Weyl (λ) or Dynkin notation (a) as

(λ)	(a)	
$[1^x]$	$(0\ldots010\ldots0)$	$x=1,2,\ldots,k-2$
Δ_-	$(0\ldots0\ldots10)$	
Δ_+	$(0\ldots0\ldots01)$	

where a choice is made in relating Δ_+ and Δ_- to the corresponding Dynkin labels. The above choice leads to the relationship for an arbitrary irrep $[\lambda]$ of SO_{2k}

$$\begin{aligned}&\lambda_i=\sum_{j=i}^{k-2}a_j+\tfrac{1}{2}(a_{k-1}+a_k), \qquad i=1,2,\ldots,k-2,\\ &\lambda_{k-1}=\tfrac{1}{2}(a_{k-1}+a_k), \qquad \lambda_k=\tfrac{1}{2}(-a_{k-1}+a_k),\end{aligned} \tag{2}$$

and inversely

$$\begin{aligned}&a_i=\lambda_i-\lambda_{i+1}, \qquad i=1,2,\ldots,k-1,\\ &a_k=\lambda_{k-1}+\lambda_k.\end{aligned} \tag{3}$$

The highest weight irreps arising in the reduction of the fundamental irreps of SO_{2k} under $SO_{2k}\downarrow U_1\times SU_k$ are (Black and Wybourne 1983)

$$\begin{aligned}&[1^x]\supset\{x\}\{1^x\} \qquad (x\leqslant k-2),\\ &[1^k]_+\supset\{k\}\{0\}, \qquad [1^k]_-\supset\{k-2\}\{2^{k-1}\},\\ &\Delta_-\supset\{k/2-1\}\{1^{k-1}\}, \qquad \Delta_+\supset\{k/2\}\{0\}.\end{aligned} \tag{4}$$

Let us now label the irreps of SO_{2k} in terms of such highest weight irreps of the maximal subgroup $U_1\times SU_k$ by putting

$$\begin{aligned}&\nu_i=\sum_{j=i}^{k-1}a_j, \qquad \nu_k=0,\\ &\mu=\sum_{j=1}^{k}ja_j-\tfrac{1}{2}k(a_{k-1}+a_k).\end{aligned} \tag{5}$$

The irreps of SO_{2k} may now be unequivocally labelled as $[\mu;\nu]$ and under $SO_{2k}\downarrow U_1\times SU_k$ we have

$$[\mu;\nu]\supset\{\mu\}\{\nu\}$$

as the leading term.

It follows from (4) that

$$\begin{aligned}&a_i=\nu_i-\nu_{i+1}, \qquad i=1,2,\ldots,k-1,\\ &a_k=(2/k)(\mu-\omega_\nu)+\nu_{k-1},\end{aligned} \tag{6}$$

where ω_ν is the weight of the partition (ν) and hence

$$\begin{aligned} &\nu_i = \lambda_i + \lambda_{k-1} \qquad (i = 1, 2, \ldots, k-2), \\ &\nu_{k-1} = \lambda_{k-1} - \lambda_k, \qquad \nu_k = 0, \qquad \mu = \sum \lambda_i = \omega_\lambda, \end{aligned} \tag{7}$$

where ω_λ is the weight of the partition (λ). Inversely

$$\begin{aligned} &\lambda_i = \nu_i + \nu_{k-1} + (\mu - \omega_\nu)/k, \qquad i = 1, 2, \ldots, k-2, \\ &\lambda_{k-1} = \nu_{k-1} + (\mu - \omega_\nu)/k, \qquad \lambda_k = (\mu - \omega_\nu)/k. \end{aligned} \tag{8}$$

For SO_{16} we have the labelling equivalences

$$\begin{aligned} &[\Delta; 2^8]_+ \equiv (00000005) \equiv [20; 0], \qquad [\Delta; 2^8]_- \equiv (00000050) \equiv [15; 5^7], \\ &[\Delta; 32^7]_+ \equiv (10000005) \equiv [21; 1], \qquad [321] \equiv (11100000) \equiv [6; 321], \\ &[1^8]_+ \equiv (00000002) \equiv [8; 2], \qquad [1^8]_- \equiv (00000020) \equiv [6; 2^7]. \end{aligned}$$

3. Reconstruction of SO_{2k} strings from $U_1 \times SU_k$ strings

The reduction of a (possibly reducible) representation $[\lambda]$ of SO_k under $SO_{2k} \downarrow U_1 \times SU_k$ leads to a string of irreps of $U_1 \times SU_k$ which can be sequenced in order of highest weight for U_1 and then for SU_k. Thus a typical $U_1 \times SU_8$ string could be

$$\{4\}\{0\} + \{2\}(\{1^6\} + \{1^2\}) + \{0\}(\{21^6\} + \{1^4\} + \{0\}) + \{-2\}(\{1^6\} + \{1^2\}) + \{-4\}\{0\}. \tag{9}$$

A $U_1 \times SU_k$ string can be reconstructed as a SO_{2k} string by first examining the leading term in the sequenced $U_1 \times SU_k$ string and noting (1). Thus in the above string the leading term is $\{4\}\{0\} \to [4; 0] \equiv [\Delta; 0]_+$. Under $SO_{16} \downarrow U_1 \times SU_8$ we have (Black and Wybourne 1983)

$$[\Delta; 0]_+ \downarrow \{4\}\{0\} + \{2\}\{1^6\} + \{0\}\{1^4\} + \{-2\}\{1^2\} + \{-4\}\{0\}.$$

If these terms are deleted from (9) we are left with the residual string

$$\{2\}\{1^2\} + \{0\}(\{21^6\} + \{0\}) + \{-2\}\{1^6\}. \tag{10}$$

The leading term is $\{2\}\{1^2\} \equiv [2; 1^2] \equiv [1^2]$. Under $SO_{16} \downarrow U_1 \times SU_8$ $[1^2]$ decomposes into just the terms contained in (10) and hence the original $U_1 \times SU_8$ string arises from the reduction of $[\Delta; 0]_+ + [1^2]$ of SO_{16} under $SO_{16} \downarrow U_1 \times SU_8$.

The above procedure allows any $U_1 \times SU_k$ string derived from a $SO_{2k} \downarrow U_1 \times SU_k$ reduction to be uniquely reconstructed as a SO_{2k} string.

4. $E_8 \downarrow SO_{16}$ branching rules

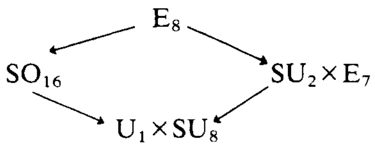
We now show how to use the preceding remarks to obtain the branching rules for $E_8 \downarrow SO_{16}$ unambiguously from those for $E_8 \downarrow SU_2 \times E_7$ and $E_7 \downarrow SU_8$ by considering the two group chains

$$E_8 \downarrow SU_2 \times [E_7 \downarrow SU_8] \downarrow U_1 \times SU_8 \tag{11a}$$

and

$$E_8 \downarrow SO_{16} \downarrow U_1 \times SU_8. \tag{11b}$$

The two groups $U_1 \times SU_8$ terminating the two chains coincide if the weights characterising the irreps of U_1 in (11) are divided by two. This factor of two arises simply in the particular choice made in deriving the $SO_{2k} \downarrow U_1 \times SU_k$ branching rule (Black and Wybourne 1983). With this in mind we can portray the group chains as



If the $E_8 \downarrow SU_2 \times E_7$ and $E_7 \downarrow SU_8$ branching rules are known then the $U_1 \times SU_8$ content of an E_8 irrep can be determined to yield a string of $U_1 \times SU_8$ irreps. This string can be reconstructed as a string of SO_{16} irreps corresponding to the SO_{16} content of the E_8 irrep.

The irreps of E_8 may be conveniently labelled either in terms of the labels used for the maximal subgroup SU_9 involving partitions (λ) into at most eight parts or in terms of those (μ) used for the SO_{16} maximal subgroup (King and Al-Qubanchi 1981a). The two sets of labels may be intraconverted by noting that

$$\begin{aligned}
\lambda_1 &= \mu_1 + \mu_2, & \mu_1 &= \lambda_1 - \tfrac{1}{6}\omega_\lambda + \tfrac{1}{2}\lambda_2,\\
\lambda_2 &= \mu_1 + \mu_2 - \tfrac{1}{2}\omega_\mu, & \mu_2 &= \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2,\\
\lambda_3 &= \mu_2 - \mu_8, & \mu_3 &= -\lambda_8 + \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2,\\
\lambda_4 &= \mu_2 - \mu_7, & \mu_4 &= -\lambda_7 + \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2,\\
\lambda_5 &= \mu_2 - \mu_6, & \mu_5 &= -\lambda_6 + \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2,\\
\lambda_6 &= \mu_2 - \mu_5, & \mu_6 &= -\lambda_5 + \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2,\\
\lambda_7 &= \mu_2 - \mu_4, & \mu_7 &= \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2 - \lambda_4,\\
\lambda_8 &= \mu_2 - \mu_3, & \mu_8 &= -\lambda_3 + \tfrac{1}{6}\omega_\lambda - \tfrac{1}{2}\lambda_2.
\end{aligned} \tag{12}$$

Use of $E_8 \downarrow SU_2 \times E_7$ and $E_7 \downarrow SU_8$ branching rules given by Wybourne and Bowick (1977) and Wybourne (1979) readily verified the $E_8 \downarrow SO_{16}$ decompositions given by King and Al-Qubanchi (1981b) in an unambiguous manner. Their results may be extended either by exploiting the E_8 Kronecker products given by Wybourne (1979) together with those of SO_{16} or by extending the $E_8 \downarrow SU_2 \times E_7$ branching rules and following the above procedure.

King *et al* (1981) and Black *et al* (1983) have given systematic procedures for resolving Kronecker products and up to fourth powers of irreps of SO_{2k}. Consider the 4881384-dimensional irrep of E_8 labelled (42) in the SU_9 scheme and (4) in the SO_{16} scheme. Inspection of Wybourne's E_8 tables (1979) shows that in the SO_{16} scheme

$$(2) \otimes \{2\} = (4) + (31^3) + (2^2) + (2) + (0) + (\Delta; 2)_+. \tag{13}$$

The $E_8 \downarrow SO_{16}$ branching rules for all the irreps, apart from (4), appearing on the right-hand side of (13) are given by King and Al-Qubanchi (1981b). Furthermore, under $E_8 \downarrow SO_{16}$ we have

$$(2) \downarrow [2] + [1^4] + [\Delta; 1]_-. \tag{14}$$

Thus the SO_{16} content of (13) follows upon evaluating the plethysm (Wybourne 1970)

$$([2]+[1^4]+[\Delta;1]_-)\otimes\{2\}$$
$$=[2]\otimes\{2\}+[1^4]\otimes\{2\}+[\Delta;1]_-\otimes\{2\}$$
$$+[2]\cdot[1^4]+[2]\cdot[\Delta;1]_-+[1^4]\cdot[\Delta;1]_-. \qquad (15)$$

The SO_{16} plethysms and products may be evaluated as in King *et al* (1981) and Black *et al* (1983) to yield a string of SO_{16} irreps. The SO_{16} irreps associated with the $(31^3)+(2^2)+(\Delta;2)_++(2)+(0)$ are removed from the string to leave the SO_{16} content of the (4) irrep of E_8. A similar examination of the antisymmetric part of the Kronecker square of the (2) irrep of E_8 yields the SO_{16} string associated with the $E_8\downarrow SO_{16}$ decomposition of the $(\Delta;3)_-$ irrep of E_8.

The corresponding branching rule for the (42) irrep of E_8 follows from the use of the E_8 Kronecker product

$$(2)\times(2^2)=(42)+(321)+(31^3)+(31)+(2^2)+(21^2)+(2)+(\Delta;31)_++(\Delta;2)_+.$$

Likewise the (41^4) irrep of E_8 arises in the antisymmetric part, $(1^2)\otimes\{1^4\}$, of the fourth power of the (1^2) irrep of E_8. The evaluation of the relevant SO_{16} plethysms follows from King *et al* (1981). The branching rule for the (421^2) irrep of E_8 may then be found by noting that

$$(31^3)\times(1^2)-(21^2)\times(2)=(421^2)+(41^4)+(31)+(2^2)+(21^2)+(2)+(1^2).$$

Continuing in this manner, it is not difficult to deduce many further $E_8\downarrow SO_{16}$ branching rules. By way of examples we give in table 1 the branching rules for four non-trivial

Table 1. $E_8\downarrow SO_{16}$ branching rules.

$D_{(\lambda)}$	E_8	SO_{16}
4 881 384	(4)	[4]+[31111111]−+[3111]+[2222]+[221111]+[22] +[21111]+[11111111]++[1111]+[0]+[s; 3]−+[s; 2111]− +[s; 21]−+[s; 1111]++[s; 11]++[s; 0]+
6 696 000	$(\Delta;3)_-$	[s; 3]−+[s; 2111]−+[s; 211]++[s; 21]−+[s; 2]+ +[s; 11111]−+[s; 111]−+[s; 11]++[s; 1]−+[311111] +[3111]+[31]+[22211]+[22111111]−+[2211] +[2111111]+[21111]+[211]+[111111]+[11]
26 411 008	$(\Delta;31)_+$	[s; 31]++[s; 221]−+[s; 21111]++[s; 2111]−+2[s; 211]+ +2[s; 21]−+2[s; 2]++[s; 1111111]−+[s; 11111]− +[s; 1111]++2[s; 111]−+[s; 11]++2[s; 1]−+[32111]+[321] +[31111111]++[311111]+[3111]+[31]+[2221111] +[22211]+[222]+[22111111]−+[221111]+[2211]+2[2111111] +2[21111]+2[211]+[2]+[11111111]−+[111111]+[1111]
76 271 625	(41^2)	[411]+[3221]+[3211111]+[32111]+[321]+[31111111]− +2[311111]+[3111]+[31]+[222211]+[2221111] +[22211]+[22111111]−+[22111111]++2[221111] +3[2211]+[22]+2[2111111]+2[21111]+2[211] +[11111111]++2[111111]+[1111]+[11]+[s; 311]− +[s; 31]++[s; 3]−+[s; 2211]++[s; 221]−+[s; 22]+ +[s; 211111]−+[s; 21111]++2[s; 2111]−+2[s; 211]+ +3[s; 21]−+[s; 2]++[s; 111111]++[s; 11111]− +2[s; 1111]++2[s; 111]−+3[s; 11]++[s; 1]−+[s; 0]+

irreps of $E_8 \downarrow SO_{16}$. It is important to note that the methods used here yield the required results unambiguously. The first branching rules are established using the known $E_8 \downarrow SU_2 \times E_7$ and $E_7 \downarrow SU_8$ branching rules. Once these are unambiguously established we can then use E_8 Kronecker products to produce new $E_8 \downarrow SO_{16}$ branching rules without resorting to dimensional or Dynkin index methods that are ambiguous.

The branching rules evaluated here were rapidly calculated using the program SCHUR (Black 1983) to compute the E_8 and SO_{16} Kronecker products, and the $SO_{16} \downarrow U_1 \times SU_8$ branching rules. The $E_8 \downarrow SO_{16}$ branching rules were built up in SCHUR which checked the final results by computing the dimensions and Dynkin index for the E_8 and SO_{16} irreps.

5. Conclusion

Methods for evaluating $E_8 \downarrow SO_{16}$ branching rules in an unambiguous manner have been outlined and applied to some non-trivial examples. An alternative method of labelling the irreps of SO_{2k} has been developed and exploited. It remains to be seen whether such a labelling scheme will lead to significant simplifications in computing properties of the SO_{2k} groups.

Acknowledgments

This work was stimulated in part by a preprint from G Bélanger of Johns Hopkins University and was greatly assisted by G R E Black's SCHUR program. I am grateful to Dr R C King for a number of useful comments.

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